Enrollment No: ______________________ **Exam Seat No:**______________________

C.U.SHAH UNIVERSITY

Winter Examination-2022

Subject Name: Advanced Functional Analysis

Subject Code: 5SC04AFA1 **Branch: M.Sc. (Mathematics)**

Semester: 4 **Date: 19/09/2022** **Time: 02:30 To 05:30** **Marks: 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **[07]**

a. Define Hilbert space. (01)

b. Let X be an inner product space and $\{x_1, x_{2,} \dots, x_n\}$ be orthogonal set in X. Prove that $\|x_1 + x_2 + \dots + x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \dots + \|x_n\|^2$. (02)

c. State Bessel's inequality. (02)

d. State and prove Parallelogram Law. (02)

Q-2 **Attempt all questions** **[14]**

a. State and prove Schwarz Inequality. (06)

b. Prove that l^p is an inner product space if and only if $p = 2$. (05)

c. Let X be an inner product space and $\{x_1, x_{2,} \dots\}$ be orthogonal set in X. Then prove that $\|x_1 + x_2 + \dots + x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \dots + \|x_n\|^2$. (03)

OR

Q-2 **Attempt all questions** **[14]**

a. Let X be an inner product space and E be an orthonormal subset of X. Then prove the following (08)

(i) For each $x \in X$, $E_x = \{u \in E : < x, u > \neq 0\}$ is countable.
(ii) If $E_x = \{u_1, u_2, \dots\}$ then $< x, u_n > \to 0$ as $n \to \infty$.
(iii) If X is a Hilbert space and $\sum_{n=1}^{\infty} < x, u_n > u_n$ converges to $y \in X$ then $(x - y) \perp E$.

b. If H be Hilbert space, $\{\alpha_n\}$ be a sequence in K and $\{u_{1,} u_{2,} \dots\}$ be an orthonormal subset of H then prove that $\sum_{n=1}^{\infty} \alpha_n u_n$ converges in H if and only if $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$. In this case $\alpha_n = < x, u_n >$ for all n. (06)

Q-3 **Attempt all questions** **[14]**

a. State and prove unique Hahn – Banach extension theorem. (08)

b. Let X be an inner product space. Y be a subspace of X and $x \in X$. Then show that $y \in Y$ is a best approximation from Y to x id and only if $(x - y) \perp Y$. (06)

OR

Q-3 **Attempt all questions** **[14]**

a. State and prove Projection Theorem. (08)

b. Let H be a Hilbert space and X be a subspace of H. Let $g \in X'$. Then show that there exists a unique $f \in H'$ such that $f|_X = g$ and $||f|| = ||g||$. (06)

SECTION – II

Q-4 **Attempt the Following questions** **[07]**

a. Let H be a Hilbert space and $T, S \in BL(H)$. Prove that (01)

$$(S + T)^* = S^* + T^*.$$

b. Define Compact Operator and Normal Operator. (02)

c. Let H be a Hilbert space and $T \in BL(H)$ be normal. If $x \in H$ such that $(T - \lambda I)^2 x = 0$ then prove that $(T - \lambda I)x = 0$. (02)

d. Let H be a Hilbert space and $T \in BL(H)$. Prove that T is isometry if $T^*T = I$. (02)

Q-5 **Attempt all questions** **[14]**

a. Let H be a Hilbert space and $T \in BL(H)$. Then prove that T is bounded below if and only if $R(T^*) = H$. (07)

b. Let H be a Hilbert space and $T \in BL(H)$. Prove the following statements: (04)

a) $\ker(T) = R(T^*)^{\perp}$

b) T is one-one if and only if $\overline{R(T^*)} = H$.

c. Prove that the set of all bounded self - adjoint operator on a Hilbert space is closed in $BL(H)$. (03)

OR

Q-5 **Attempt all questions** **[14]**

a. Let H be a Hilbert space and $T \in BL(H)$. Then show that there is a unique $S \in BL(H)$ such that $< Tx, y > = < x, Sy >$ for every $x, y \in H$ and $||S|| \leq ||T||$. (07)

b. Let H be a Hilbert space and $S, T \in BL(H)$. Prove the following: (07)

(i) Let S and T be normal. If S commutes with T^* and T commutes with S^* then $S + T$ and ST are normal.

(ii) Let S and T be self-adjoint. Then $S + T$ is self-adjoint. Also ST is self adjoint if and only if S and T commutes.

Q-6 **Attempt all questions** **[14]**

a. Let H be a Hilbert space and $T \in BL(H)$ be normal. Then prove that the eigen vectors corresponding to distinct eigen values of T are orthogonal. Does the result hold if T is not normal ? Justify. (08)

b. Define Numerical Range of $T \in BL(H)$. Prove (i) $\sigma_a(T) \subset \overline{W(T)}$ and (ii) $\sigma(T) \subset \overline{W(T)}$. (06)

OR

Q-6 **Attempt all Questions** **[14]**

a. Let H be a Hilbert space and $T \in BL(H)$. Then prove the following (08)

(i) $\sigma(T) = \{\overline{\lambda} : \lambda \in \sigma(T^*)\}$

(ii) $\sigma_e(T) \subset \sigma_a(T)$ and $\sigma(T) = \sigma_a(T) \cup \{\overline{\lambda} : \lambda \in \sigma_e(T^*)\}$.

b. Let H be a Hilbert space and $T \in BL(H)$ be compact then show that T^* is compact. (06)